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REMARKS ON THE POSITIVITY OF DENSITIES OF STABLE PROBABILITY MEASURE ON R^d

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Abstract. Let μ be an index α, $0 < \alpha < 2$, stable prob. measure on R^d, the d-Euclidean space. Let σ be the spectral measure of μ on the boundary of the unit sphere of R^d; and assume that the support of σ is d-dimensional. Using known results about the support of μ, simple proofs are provided for the following two facts about the continuous bounded density f_μ of μ : (i) If $1 \leq \alpha < 2$, then f_μ is positive on R^d; (ii) if $0 < \alpha < 1$, then $f_\mu(x) > 0$ if and only if x belongs to the interior of the translated cone $a_0 + C_0$, where C_0 is the smallest closed cone generated by the support of σ, and a_0 is the centering element of μ.

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REMARKS ON THE POSITIVITY OF DENSITIES OF STABLE PROBABILITY MEASURE ON R^d

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1. Introduction and Preliminaries:. Let μ be an index α, $0 < \alpha < 2$, stable prob. measure on R^d, the d-Euchilidian space. Let σ be the spectral measure of μ on the boundry of the unit sphere of R^d; and assume that the support of σ is d-dimensional. Using known results about the support of μ, simple proofs are provided for the following two facts about the continuous bounded density f_μ of μ : (i) If $1 \leq \alpha < 2$, then f_μ is positive on R^d; (ii) if $0 < \alpha < 1$, then $f_\mu(x) > 0$ if and only if x belongs to the interior of the translated cone $a_0 + C_0$, where C_0 is the smallest closed cone generated by the support of σ, and a_0 is the centering element of μ. Both of these results are known, we learned this from the unpublished manuscript of Kesten [**4**]. In this paper, he showed that if x belong to the interior of $a_0 + C_0$ then $f_\mu(x) > 0$; the fact that f_μ vanished on the complement of $a_0 + C_0$ was shown earlier by Taylor [**9**]. Kesten also proved a slightly weaker result than (ii) for the case $1 < \alpha < 2$; however, Fristedt mentioned in [**3**] that (ii) can be recovered from Prot [**5**] and Taylor [**9**]. All these proofs seem quite lengthy and use, in a substential way, the representation theory of Lévy processes. Kesten [**4**] asked if (i) and (ii) can be proved using only the Fourier analytic methods. Our proof of (i) and (ii) use (via Theorem 2.1) certain results from the theory of characterstic functions of measures on R^d and also from the theory of weak convergence; thus, though our proofs are not entirely based on Fourier analytic methods, they seem more direct and simpler. In addition, we also consider the analogs of (i) and (ii), when σ is not d-dimensional; further, we discuss the orthogonality and equivalence dichotomy for two stable probability measures on R^d (Theorem 2.6).

In the rest of this section we include necessary notation and some difinitions.

Let ν be a finite measure on the Borel $\sigma-$algebra of a separable metric space B, then S_ν, the support of ν, is the smallest closed set with full ν-measure. It is easy to see that

$$S_\nu = \{x \epsilon B : \nu(V_x) > 0, \text{ for every open } nbd\, V_x \text{ of } x\}.$$

For a given measure ν on B, S_ν will denote, throughout, the support of ν; further, for any set $A \subseteq B$, $\overline{A}$ and $int(A)$ will denote the closure and the interior of A, repectively. For an $\varepsilon > 0$ and $x \epsilon B$, $\Delta(x, \varepsilon)$ will denote the set $\{y \epsilon B, ||y - x|| < \varepsilon\}$. Finally, for a set A in R^d, $sp(A)$ and $cc(A)$ will denote the smallest linear space and the smallest convex cone generated by A, respectively.

Let μ be an index $0 < \alpha < 2$ stable probability measure on R^d. Then, as in well known, for any $0 < \tau < \infty$, μ can be written as

$$\mu = \delta(a_\tau) * \mu_\tau \tag{1.1}$$

,where $\delta(a_\tau)$ denotes the Dirac measure at $a_\tau(\mu) = a_\tau \epsilon R^d$ and μ_τ is the index α stable probability measure with the characterstic (ch.) function

$$\hat{\mu}_\tau(y) = \exp \int_{\partial\Delta_1} \int_0^\infty \{e^{i<u,y>} - 1 - it \underset{[0,\tau]}{\mathrm{I}(t)} < u, y >\} \frac{dt}{t^{1+\alpha}} \sigma(du), \tag{1.2}$$

$y \epsilon R^d$, where $\Delta_1 = \Delta(0,1)$, $\partial\Delta_1$ is the boundry of Δ_1 and σ is a finite measure on $\partial\Delta$, (the measure σ is refered to as the *spectral measure* μ; this measure can and will be assumed symmetric, if μ is symmetric. Further, if $0 < \alpha < 1$ (resp. $1 < \alpha < 2$), then one can write

$$\mu = \delta(a_0) * \mu_0 \qquad (\text{resp. } \mu = \delta(a_\infty) * \mu_\infty), \tag{1.3}$$

where $a_0, a_\infty \in R^d$ and the ch. function of μ_0 (resp. of μ_∞) is given by (1.2) with Δ_τ replaced by $\Delta_0 = \{0\}$ (resp. by $\Delta_\infty = R^d$). The prob. measure μ_0 and μ_∞ are index α ***strictly stable components*** of μ in these two cases. For a given index α stable prob. measure μ on R^d, the notation a_τ, μ_τ, will always be used to denote the elements and the measures introduced in (1.1) and (1.2).

2. The positivity of the density of stable measures on R^d.

We begin by stating a result for the support of stable prob. measures on R^d. Proofs of part (a) can be found in **[1,10]** and that of part (b) in **[8]**; the proofs of these two results in the symmetric case were given earlier in **[6,7]**, for all $0 < \alpha < 2$; and, for the case $1 < \alpha < 2$, in **[2]**.

THEOREM 2.1. *Let μ be an index $0 < \alpha < 2$ prob. measure on R^d; and let $\delta(a_\tau)$ and μ_τ be the component measures of μ as in (1.1). Then we have the following:*

(a) If $0 < \alpha < 1$, then

$$\mu_\tau = \delta(b_\tau) * \mu_0, \quad S\mu_\tau = b_\tau + \overline{cc}(S_\sigma) \tag{2.1}$$

for every $0 \le \tau < \infty$, where $b_\tau = \left(\int_{\partial V} u d\sigma\right)\left(\frac{\tau^{1-\alpha}}{\alpha_1}\right)$ and which belongs to $\overline{sp}(S_\sigma)$; further, $S_{\mu_\tau}\overline{cc}(S_\sigma) = \overline{sp}(S_\sigma)$ for one and hence all $0 \le \tau < \infty$ if and only if $S_{\mu_\tau} oy^{-1} = R$, for every $y \in R^d$, for which $\mu_\tau oy^{-1}$ is a non-degenerate measure on R.

(b) If $1 \le \alpha < 2$, then

$$S\mu_\tau = \overline{sp}(S_\sigma) \tag{2.2}$$

,for every $0 < \tau < \infty$; further, if $1 < \alpha < 2$, then

$$\mu_\tau = \delta(c_\tau) * \mu_\infty$$

, *for every* $0 < \tau \leq \infty$, *where* $c_\tau = (\int_{\partial U} u d\sigma)(\frac{\tau^{1-\alpha}}{1-2})$ *and which belong to* $\overline{sp}(S_\sigma)$ *(here* $c_\infty \equiv 0$*), therefore, in this case, (2.2) is valied for* $\tau = \infty$ *as well.*

REMARK 2.2: Note that it follows immediately, from (1.1), (2.1) and (2.2) that

$$S_\mu = a_\tau + \overline{sp}(S\sigma)\,, \tag{2.3}$$

if $1 \leq \alpha < 2$; and

$$S_\mu = a_\tau + b_\tau + \overline{cc}(S_\sigma), \tag{2.4}$$

if $0 < \alpha < 1$. Further, if μ is symmetric, then if follows, from (2.2), (2.3), the symmetry of σ, and the facts that $a_\tau = b_\tau = \theta$, that

$$S_\mu = \overline{sp}(S_\sigma) \tag{2.5}$$

The following lemma shows that every measure μ_τ (see (1.2)), when restricted to a suitable subspace of R^d, has a bounded continuous density, this fact seems to be known for quite some time. The folloiwng proof of this lemma is due to Kesten [4]; and included here for completeness.

We will use the following additional notation throughout the note. If σ is a finite measure on $\partial\Delta_1$, then we shall denote $\overline{cc}(S_\sigma)$ and $\overline{sp}(S_\sigma)$, respectively, by $C_0(\sigma)$ and $E_0(\sigma)$. Further, we shall supress σ from these notations, whenever there is no likely confusion.

LEMMA 2.3. *Let* μ_τ *be the index* α *stable prob. measure on* R^d *with the ch. function* $\hat{\mu}_\tau$ *given by (1.2), where it is assumed that* τ *can take value 0 (resp.* ∞*), if* $0 < \alpha < 1$ *(resp.* $1 < \alpha < 2$*). Then* $\mu_\tau(E_0(\sigma)) = 1$ *and the ch. function of* μ_τ*(restricted to* $E_0(\sigma)$*) satisfies*

$$|\hat{\mu}_\tau(y)| \leq e^{-K\|y\|^\alpha} \tag{2.6}$$

, *for every* $y \in E_0(\sigma)$, *where* K *is the real positive constant given by*

$$K = \left[\inf_{v \in \partial\Delta_1 \cap E_0(\sigma)} \int_{S_\sigma \cap E_0(\sigma)} | < u, v > |^\alpha \, \sigma(du)\right] \left[\int_0^\infty \left(\frac{1 - \cos s}{s^{1+\alpha}}\right) ds\right].$$

PROOF: That $\mu_\tau(E_0) = 1$, follows from (2.1) and (2.2).

Clearly, for $y \in E_0$, we have

$$|\hat{\mu}_\tau(y) = \exp\left[\int_{S_\sigma \cap E_0} \{(\cos t < u, y > -1)\frac{dt}{t^{1+\alpha}}\}\sigma(du)\right]. \tag{2.7}$$

Fix $y \in E$, $y \neq 0$, (for $y = 0$, (2.6) is obvious), then making the change of variable $t| < u, y > | = s$ in (2.7), one obtains

$$\log|\hat{\mu}_\tau(y) = \|y\|^\alpha \left[\int_{A_y} | < u, \frac{y}{\|y\|} > |^\alpha \sigma(du)\right] \left[\int_0^\infty \left(\frac{\cos s - 1}{s^{1+\alpha}}\right) ds\right], \tag{2.8}$$

where $A_y = \{u \in E :< u, y > \neq 0\}$. Next noting that $\sigma(A_y) > 0$ (otherwise $\{u \in E_0 :< u, y >= 0\} \cap S_\sigma$ will have a full σ-measure and $\overline{sp}(S_\sigma) \subseteq \{y \in E_0 :< u, y >= 0\}$; which will contradict the fact $\overline{sp}(S\sigma) = E_0$), one observes that

$$\psi(v) = \int_{S_\sigma \cap E_0} | < u, v > |^\alpha \sigma(du)$$

is positive on $\partial\Delta_1 \cap E_0$ and, as it is clearly continuous and $S_\sigma \cap E_0$ is a compact set, we have

$$\int_{A_y} | < u, \frac{y}{\|y\|} > |^\alpha \sigma(du) \geq \inf_{v \in \partial\Delta_1 \cap E_0} \psi(v) = \psi(v_0) > 0,$$

for some $v_0 \in \partial\Delta_1 \cap E_0$. Therefore, from (2.8),

$$\log|\hat{\mu}_\tau(y)| \geq -\|y\|^\alpha \psi(v_0) \int_0^\infty \left[\frac{1 - \cos s}{s^{1+\alpha}}\right] ds,$$

which proves (2.7).

The above lemma immediately yields the following corollary.

COROLLARY 2.4: Let μ_τ be the measure in Lemma 2.3; then μ_τ restricted to $E_0(\sigma)$ has a continuous bounded density; (we shall denote this density, throughout by f_τ).

Now we are ready to prove the main result of this note.

THEOREM 2.4. *Let μ be an index α stable prob. measure on R^d with spectral measure σ. Then we have the following:*

(a) If $0 < \alpha < 1$, then, for every $0 \leq \tau < \infty$, the (bounded continuous) density f_τ of the measure μ_τ (see (2.1)) restricted to $E_0(\sigma)$ is positive on the interior (in E) of the translated cone $C_\tau \equiv b_\tau + C_0(\sigma)$ and zero on $E \backslash C_\tau$; further, for every $0 \leq \tau < \infty$,

$$f_\tau(x) = f_0(x - b_\tau) \tag{2.9}$$

, for every $x \in E_0(\sigma)$.

(b) If $1 \leq \alpha < 2$, then, for every $0 < \tau < \infty$, the (bounded continuous) density f_τ of μ_τ (restricted to $E_0(\sigma)$) is positive on $E_0(\sigma)$; further, if $1 < \alpha < 2$, the same is true of f_∞ and, in this case, for every $0 < \tau \leq \infty$,

$$f_\tau(x) = f_\infty(x - (c_\tau))$$

, for every $x \in E_0(\sigma)$.

PROOF OF (a): From (2.1), we already know that (2.9) is valied. Next observe, by the continuity of f_τ and the fact $S_{\mu_\tau} = C_\tau$, that $f_\tau = 0$ on $E \backslash C_\tau$. Thus, noting that $x \in Int(C_\tau)$ if and only if $x - b_\tau \in Int(C_0)$(note $C_0 = \overline{cc}(S_\sigma)$, as $b_0 = 0$), the proof of (a) will be complete if we can prove that f_0 is positive on $Int(C_0)$. We prove this in the following:

Let $x \in Int(C_0)$; if $x = 0$ then clearly $C_0 = E_0$ and one can use the argument of part (b) to show that $f_0(0) > 0$. So we assume $x \neq 0$, and set $x_0 = 2^{1/\alpha}x$; then, since x_0 is also an interior point of C_0, we can find an $\varepsilon > 0$ such that $\Delta(x_0, \varepsilon) \subseteq C_0$. Now let $x_1 = (\frac{\varepsilon}{4}) 4 \frac{x_0}{\|x_0\|}$ and let $0 < \varepsilon' < \frac{\varepsilon}{4}$ be such that $\Delta(x_1, \varepsilon') \subseteq C_0 = S_{\mu_0}$. It follows, using the continuity of f_0, that there exists a $y_1 \in \Delta(x_1, \varepsilon')$ and $0 < \varepsilon_1 < \varepsilon'$ such that

$$\Delta(y_1, \varepsilon_1) \subseteq \Delta(x_1, \varepsilon') \text{ and } f_0 > 0 \text{ on } \Delta(y_1, \varepsilon_1). \tag{2.10}$$

Next observe that $\Delta(x_0 - y_1, \varepsilon_1) \subseteq \Delta(x_0, \varepsilon) \subseteq C_0 = S_{\mu_0}$; in face, if $z \in \Delta(x_0 - y_1, \varepsilon_1)$, then

$$\|z - x_0\| \leq \|z - (x_0 - y_1)\| + \|y_1 - x_1\| + \|x_1\| \leq \frac{3}{4}\epsilon$$

(recall $\varepsilon_1 < \frac{\varepsilon}{4}$). Hence, again using continuity of f_0, we can find *a* $z_0 \in \Delta(x_0 - y_1, \epsilon_1)$ and $0 < \varepsilon_2 (< \varepsilon_1)$ such that

$$\Delta(z_0, \varepsilon_2) \subseteq \Delta(x_0 - y_1, \varepsilon_1) \text{ and } f_0 > 0 \text{ on } \Delta(z_0, \varepsilon_2). \tag{2.11}$$

Now $\Delta(z_0, \varepsilon_2) \subseteq \Delta(x_0 - y_1, \varepsilon_1)$ clearly implies that $\Delta(x_0 - z_0, \varepsilon_2) \subseteq \Delta(y_1, \varepsilon_1)$; therefore, since $y \in \Delta(z_0, \varepsilon_2)$ if and only if $x_0 - y \in \Delta(x_0 - z_0, \varepsilon_2)$, it follows, from (2.10) and (2.11), that

$$f_0(x_0 - y) f_0(y) > 0, \quad \text{for every} \quad y \in \Delta(z_0, \varepsilon_2) \tag{2.12}$$

Now using (2.12) and the following (which is a direct consiquence of strict stability of μ_0)

$$f_0 * \underset{0}{f}(\cdot) = 2^{-\frac{1}{\alpha}} f_0(2^{-\frac{1}{\alpha}} \cdot),$$

we get

$$2^{-\frac{1}{\alpha}} f_0(x) = \int_{E_0} f_0(x_0 - y) f(y) dy \geq \int_{\Delta(0, \varepsilon_2)} f_0(x_0 - y) f_0(y) dy > 0.$$

(recall $x_0 = 2^{1\backslash\alpha_x}$).

PROOF OF (b): The basic idea of the proof here is similar to (b); if fact, this case is simpler, because S_{μ_τ} is a linear space. We give an outline of the proof for $\alpha = 1$ First note that f_τ satisfies (using stability property of μ_τ)

$$f_\tau * f_\tau(\cdot) = 2^{-1} f_\tau(2^{-1}(\cdot - a))\,, \tag{2.13}$$

for some $a \in E_0$. Now let $x \in E$: set $s_0 = 2x + a$. Then from (2.13), one has

$$f_\tau * f_\tau(x_0) = 2^{-1} f_\tau(x)\,. \tag{2.14}$$

Since $S_{\mu_\tau} = E_0$ and f_τ is continuous on E_0, there exists an $x_1 \in E$ such that $f_\tau > 0$ on $\Delta(x_1, \varepsilon)$ for some $\varepsilon > 0$. For the same reasons, we can find $z_0 \in \Delta(x_0 - x_1, \varepsilon_1)$ such that $f_\tau > 0$ on $\Delta(z_0, \varepsilon_2) \subseteq \Delta(x_0 - x_1, \varepsilon_1)$ for some $\varepsilon_2 > 0$. Then, clearly

$$f_\tau(x_0 - y) f_\tau(y) > 0 \quad \text{for all} \quad y \in \Delta(z_0, \varepsilon_2)$$

; and, it follows, from (2.14), that $f_\tau(x) > 0$.

Completing the proof.

REMARK 2.5: (a) Let μ be as in Theorem 2.4; then it follows from the definition of support and the theorem that μ (restricted to some subspace of R^d) has a density f_μ if and only if a_τ (see (1.1)) belongs to E_0; and in the case when $a_\tau \in E_0$, μ (restricted to E_0) has a (unique) bounded continuous density f_μ on E_0. In fact, for any fixed $0 < \tau < \infty$, $f_\mu(x) = f_\tau(x - a_\tau)$, for all $x \in E_0$ (where recall f_τ is the bounded continuous density of μ_τ). Thus, if $0 < \alpha < 1$, f_μ is positive on $Int(a_\tau + b_\tau + C_0)$ and zero on the complement of this set; and if $1 \le \alpha < 2$, f_μ is positive on E_0.

(*b*) If μ is symmetric, then it follows from (*a*) above and Remark 2.2 that μ (restricted to E_0) has a (unique) bounded continuous density which is positive on E_0.

(*c*) Let μ_0 be the index $0 < \alpha < 1$ strictly stable prob. measure on R^d as in Theorem 2.1 (a), and f_0 be the bounded continuous density of μ_0 restricted to E_0 (see Theorem 2.4 (a)). Then, as noted in Section 1, it was shown by Kesten [4] that $f_0 > 0$ on $Int(cc(S_\sigma))$, and by Taylor [9] that $f_0 = 0$ on $E_\sigma \backslash Int(cc(S_\sigma))$. Since for a convex set $A \subseteq E_0$, $Int(A) = Int(\overline{A})$, these two results and the corresponding result for f_0 proved in Theorem 2.4 (a) are precisely the same.

Recall that two prob. measures on R^d are called equivalent ($\sim$) if they are mutually absolutely continuous; and they are called singular ($\perp$) if they are concentrated on two disjoint sets. The following theorem shows that two stable measure in R^d, satisfying a mild hyposthesis, are either $\sim$ of $\perp$.

THEOREM 2.6. *Let $0 < \alpha, \beta < 2$; and let μ and ν be two stable prob. measures on R^d with indices α and β, and spectral measures σ_μ and σ_ν, respectively. Assume $C_0(\sigma_\mu) = E_0(\sigma_\mu)$, if $0 < \alpha < 1$, and assume the same hypothesis for $C_0(\sigma_\nu)$, if $0 < \beta < 1$. Then either $\mu \perp \nu$ or $\mu \sim \nu$; and $\mu \sim \nu$ if and only if $E_0(\sigma_\mu) = E_0(\sigma_\nu)$.*

PROOF: Using (1.1) with $\tau = 1$, write $\mu = \delta(a_1(\mu)) * \mu_1$ and $\nu = \delta(a_1(\nu)) * \nu_1$; and recall, from (2.1) and (2.2), that

$$S_\mu = a_1(\mu) + E_0(\sigma_\mu), \qquad S_\nu = a_1(\nu) + E_0(\sigma_\nu)$$

and

$$S_{\mu_1} = E_0(\sigma_\mu),\ S_{\nu_1} = E_0(\sigma_\nu).$$

Now either $S_\mu \neq S_\nu$ or $S_\mu = S_\nu$. We will show that if the first alternative holds then $\mu \perp \nu$ and if the second alternative holds then $\mu \sim \nu$.

Clearly, if $S_\mu \neq S\nu$, then either one of these two sets is properly contained in the other or their intersection must be properly contained in both S_μ and S_ν. Suppose the first possibility holds and say $S_\nu \subseteq S_\mu$; then $a_1(\nu) + E_0(\sigma_\nu) \subseteq a_1(\mu) + E_0(\sigma_\mu)$; and, hence $a_1(\nu) - a_1(\mu) + E_0(\sigma_\nu)$ is a translate of a *proper* subspace of E. Thus, since μ_1 (restricted to $E_0(\sigma_\mu)$) has a density, $\mu_1(\{a_1(\nu) - a_1(\mu) + E_0(\sigma_\nu)\} = 0$; i.e., $\mu\{a_1(\nu) + E_0(\sigma_\nu)\} = 0$. Therefore $\mu\{S_\mu \backslash S_\nu\} = 1$ and $\nu(S_\nu) = 1$, and $\mu \perp \nu$. Under the second possibility, $S_\mu \cap S_\nu \subsetneq S_\mu$, and $S_\mu \cap S_\nu \subsetneq S_\nu$. Now

$$(2.15) \qquad [a_1(\nu) + E_0(\sigma_\nu)] \cap [a_1(\mu) + E_0(\sigma_\mu)] = a + E_0(\sigma_\nu) \cap E_0(\sigma_\mu)$$

where a is any element belonging to the left side of (2.15). Thus, $a - a_1(\mu) + E_0(\sigma_\nu) \cap E_0(\sigma_\mu) \subsetneq E_o(\sigma_\nu)$. Therefore, as before, $\mu_1[a - a_1(\mu) + E_0(\sigma_\nu) \cap E_0(\sigma_\mu)] = \mu[a + E_0(\sigma_\nu) \cap E_0(\sigma_\mu)] = 0$ and $\nu[a + E_0(\sigma_\nu) \cap E_0(\sigma_\mu)] = 0$ Hence $\mu(S_\mu \backslash S_\mu \cap S_\nu) = 1$, $\nu(S_\nu \backslash S_\mu \cap S_\nu) = 1$, and again $\mu \perp \nu$.

If $S_\mu = S_\nu S$, then $E_0(\sigma_\mu) = E_0(\sigma_\nu) = E$ (say); and, since μ_1 and ν_1 restricted to E have positive density on E by Theorem 2.4, it follows $\mu_1 \sim \nu_1$ on E; hence $\mu_1 \sim \nu_1$ on R^d. Now let A be any Borel set of R^d with $\mu(A) = 0$; then, since $\mu(A) = \mu_1(A - a_1(\mu))$ and $\mu_1 \sim \nu_1$, we have $\nu_1(A - a_1(\mu)) = 0$. Nov observing that $A - a_1(\mu) = A - a_1(\nu) + (a_1(\nu) - a_1(\mu))$ and that $a_1(\nu) - a_1(\mu) \in E$, we have

$$(2.16) \qquad \nu_1\{(A - a_1(\nu)) \cap E + a_1(\nu) + a_1(\mu)\} = 0;$$

but, since ν_1 restricted to E is equivalent to the Leb. measure on E, it follows from (2.16) that $\mu_1(A - a_1(\nu)) = 0$ or $\nu(A) = 0$. Thus $\nu << \mu$, similarly $\mu << \nu$; completing the proof.

REMARK 2.7: (a) If μ and ν in the above therorem are symmetric, then $S_\mu = E_0(\sigma_\mu)$ and $S_\nu = E_0(\sigma_\nu)$ and hence by the above theorem either $\mu \perp \nu$ or $\mu \sim \nu$, and $\mu \sim \nu$ if and only if $S_\mu = S_\nu$.

(*b*) If $0 < \alpha, \beta < 1$ and μ and ν as in the above theorem, then, in general, even in R^1, the equivalance - singularity dichotomy for μ and ν may fail For example take μ with $S_\mu = [0, \infty)$ and $\nu = \delta_{\{1\}} * \mu$ then $S_\nu = [1, \infty)$. Similar situation can prevail, even when μ and ν are strictly stable in R^d, $d \geq 2$; in fact one can take μ with $S_\mu = \{te^{i\theta} : t \geq 0, 0 \leq \theta \leq \frac{\pi}{2}\}$ and ν with $S_\nu = \{te^{i\theta} : t \geq 0, \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}$.

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